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IMPLEMENTATION OF GRAPHS USING java.util
Part Two: Weighted Graphs, Spanning Trees, and Shortest Paths

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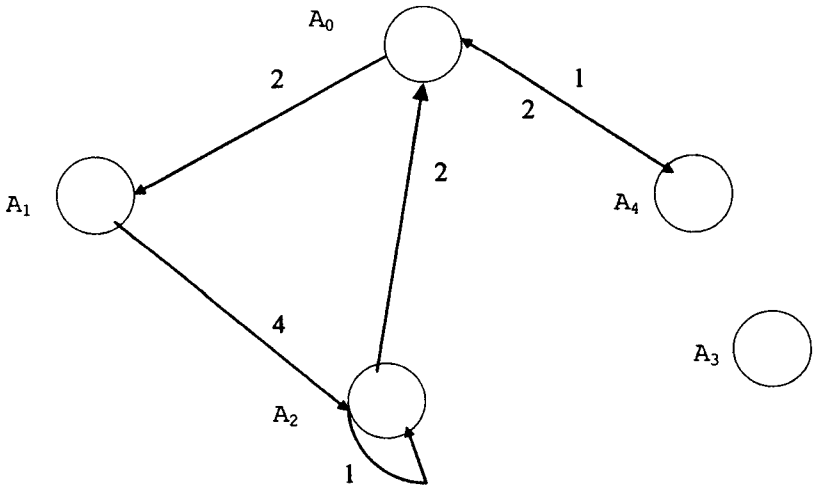
Abstract

This paper continues the discussion of the implementation of graphs using Java 5.0 begun in [1], with special emphasis on weighted graphs, both directed and undirected, as well as the treatment of minimal spanning trees and shortest paths. Again, the emphasis will be on implementing a number of the key results in this regard, using the predefined `List` interface and the `LinkedList` implementation class from the Java Collections hierarchy.

1. Weighted Graphs.

Let `G` be a digraph. If each edge of `G` has a third component whose value is a nonnegative integer, called the cost or weight of that edge, the resulting digraph is called a weighted digraph. The following is an example of a weighted digraph, where the weight `w` of each edge is represented as the third component of each member of `E`, the set of weighted edges of the weighted digraph.

Example 1: Suppose we define a weighted digraph with vertices $V = \{A_0, A_1, A_2, A_3, A_4\}$ and edges $E = \{(A_0, A_1, 2), (A_0, A_4, 1), (A_1, A_2, 4), (A_2, A_0, 2), (A_2, A_2, 1), (A_4, A_0, 2)\}$. We may illustrate this using (Figure 1).



(Figure 1)

Weighted digraphs have a number of important applications. For example, these may be used to display the mileage between cities, or the travel time from one point to another, or the cost of executing a move in a game of chess. Let P be a path from one vertex u of G to another vertex v of G. Then the weight of the path from u to v is the sum of the weights of any set of weighted edges that begin at u and terminates at v. For example, if we consider the weighted digraph described in (Figure 11.14) with $u = A_0$ and $v = A_2$, then the weight of the path from u to v may be given by the sum of the weights of the edges $(u, A_1, 2) = (A_0, A_1, 2), (A_1, A_2, 4) = (A_1, v, 4)$ or 6. However, there is another path from $u = A_0$ to $v = A_2$ given by $(u, A_1, 2) = (A_0, A_1, 2), (A_1, A_2, 4), (A_2, A_2, 1) = (A_2, v, 1)$ with weight 7, and another path given by $(u, A_1, 2) = (A_0, A_1, 2), (A_1, A_2, 4), (A_2, A_0, 2), (A_0, A_4, 1), (A_4, A_0, 2), (A_0, A_1, 2), (A_1, A_2, 4) = (A_1, v, 4)$ with weight 17. Clearly, there are many other (indeed, in this case, an infinite number of) weighted paths from $u = A_0$ to $v = A_2$.

An important and related practical problem involving weighted digraphs to solve is to determine the path between two vertices with the smallest weight. This is commonly called the Shortest Path Problem, whose solution we will discuss later in this chapter, and is given by an algorithm due to E. W. Dijkstra.[1]

For the moment, we consider the problem of how to represent a weighted digraph internally. Here we use very natural extensions of the internal representation of digraphs using adjacency matrices and adjacency lists. An adjacency matrix representation of a weighted digraph is given by a two-dimensional array of values W[i,j], where

$$W[i,j] = \begin{cases} \text{weight of the directed edge beginning at } A_i \text{ and terminating at } A_j, \text{ if such a directed edge exists;} \\ \infty, \text{ if no such edge exists.} \end{cases}$$

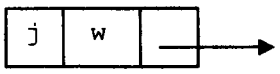
Thus the weighted digraph of (Figure 1) has the associated adjacency matrix

[1] Dijkstra, E. W., "A Note on Two Problems in Connection with Graphs," *Numerische Mathematik* 1 (1959), 269-271.

(row index)	(column index) 0	1	2	3	4
0	∞	2	∞	∞	1
1	∞	∞	4	∞	∞
2	2	∞	1	∞	∞
3	∞	∞	∞	∞	∞
4	2	∞	∞	∞	∞

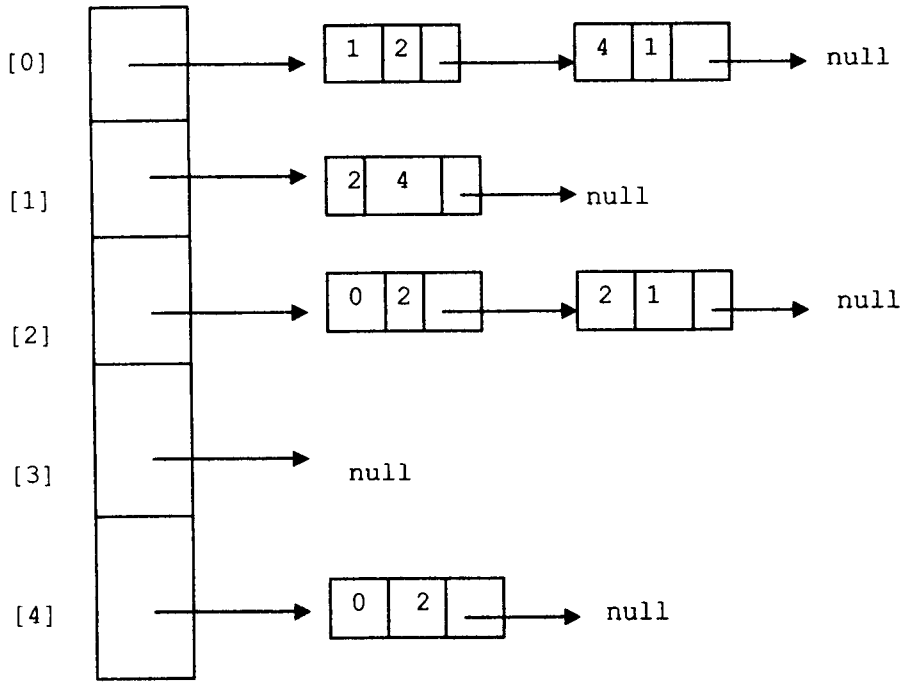
(Figure 2)

Similarly, the adjacency list representation of a weighted digraph `G` = $\{A_0, A_1, \ldots, A_{n-1}\}$ is a one-dimensional array of `n` components, each of which is a reference to a linearly linked list or to `null`. Each of the nodes on the list contains an additional component defining the weight of the directed edge. For example, if `G` contains the weighted edge (A_i, A_j, w), then component i will contain a reference to a node given by



reference to another node with the same internal structure or `null`

To illustrate this, the adjacency list of the weighted digraph of (Figure 1) is given by



(Figure 3)

How may we implement this in Java using an object-oriented design? We will define a new class called `weightedDigraph`, whose objects are weighted digraphs whose design follows the above adjacency list representation.

2. Implementation Details for Weighted Digraphs.

Following Section 7 of De Lillo[2], we will describe an implementation of weighted digraphs in Java using adjacency lists. To this end, we first describe a vertex of a weighted digraph as an object of the `weightedVertex` class, defined as

```
class weightedVertex
{
 // Instance variables.
 private int dest;
 private int weight;
 // Constructor.  Constructs a weighted vertex object whose
 // destination vertex is given by value of the first parameter
 // and whose weight is given by the value of the second
 // parameter.
 public weightedVertex(int d,int wt)
 {
  dest = d;
  weight = wt;
```

[2] De Lillo, N. J., "Implementation of Graphs Using `java.util`, Part One," submitted.

```
 } // terminates text of constructor.
 // Instance methods.

 // Retrieves weight of weighted vertex
 public int getWeight()
 {
  return weight;
 } // terminates text of getWeight()
 // Retrieves value of destination vertex
 public int getVertex()
 {
  return dest;
 } // terminates text of getVertex()
} // terminates text of weightedVertex class.
```

This implementation follows the description given in Section 1 of this paper, where each node in the list is viewed as containing an `info` component of the `weightedVertex` type. We also define an exception class used for handling exceptions that may arise using certain instance methods defined within the `weightedDigraph` class. To this end, we include the `weightedDigraphException` class, whose code is given by

```
// weightedDigraphException class
class weightedDigraphException extends RuntimeException
{
 // Constructor
 public weightedDigraphException(String str)
 {
  super(str);
 } // terminates text of constructor.
} // terminates text of weightedDigraphException class.
```

We continue the description of the implementation details by describing an interface for the `weightedDigraph` class. Again, observe that most of the instance methods described in this interface are the counterparts for weighted digraphs of corresponding instance methods of the `Digraph` class.

```
public interface weightedDigraphInterface
{
 // Tests whether the current weighted digraph is empty.
 // Returns true if so, false if not.
 public boolean isEmpty();

 // Returns the number of distinct vertices in the
 // current weighted digraph.
 public int size();

 // Retrieves sum of the weights of all of the edges of
 // the current weighted digraph.
 public int totalWeight();

 // Returns whether v is joined to w by a weighted edge.
 // Returns true if so, false if not.
 public boolean isAdjacent(weightedVertex v,weightedVertex w);
```

```
// Inserts vertex into weighted digraph.
// Inserts a new vertex if that vertex is not already present
// and raises an exception if no vertex is inserted, since it is
// already a vertex of the current weighted digraph.
public void insertVertex(weightedVertex p);

// Inserts edge from v to w.
// Constructs weighted edge from v to w if no such edge is
// currently present, and throws an exception if otherwise.
// Precondition: v,w are vertices in the current weighted digraph.
public void insertEdge(weightedVertex v,weightedVertex w);

// Removes vertex from current weighted digraph if present, along with
// all incident edges. Raises an exception if that vertex is not
// present in the current weighted digraph.
public void eraseVertex(weightedVertex v);

// Removes weighted edge from v to w if currently present in
// weighted digraph, and eliminates the weight of that edge.
// Precondition: v, w are vertices in the current weighted
// digraph.
public void eraseEdge(weightedVertex v,weightedVertex w);

// Outputs specifications of the current weighted digraph.
public void output();

} // terminates text of weightedDigraphInterface
```

We now describe the design of the instance methods and a description of the necessary instance variables of the implementation of `weightedDigraphInterface`. The instance variables, with two notable exceptions, are similar to those defined for the `Digraph` class.

```
// Gives the maximum possible number of vertices in any
// weighted digraph.
protected int ArraySize = 10;

// Number of vertices in the current weighted digraph.
protected int currSize;

// Holds the total weight of all of the edges of the current
// weighted graph.
protected int currWeight;

// Array holding the weighted vertices of the current
// weighted digraph.
protected boolean weightedVertices[] = new boolean[ArraySize];

// Tests whether the current vertex is visited in some traversal.
protected boolean isVisited[] = new boolean[ArraySize];

// Holds weighted edges in the current weighted digraph.
protected LinkedList adjList[] =
                new LinkedList[ArraySize];

// Holds vertices in the current weighted digraph.
```

```
protected Set<Integer> vertexSet = new TreeSet<Integer>();
```

The `vertexSet` instance variable holds a number of `Integer` values, none of which are repeated. All we are concerned with here is that every vertex is accounted for, so long as it appears in the weighted digraph.

The constructor constructs an initially empty `Digraph` object. This is represented in the form of an empty adjacency list, with initial size and total weight equal to zero, and containing no weighted vertices. As a result, each component of the `weightedVertices` and `isVisited` arrays is initialized as `false`. The formal code for the constructor is given by

```
// Constructor.  Constructs empty weightedDigraph object
// represented by an empty adjacency list, with current
// size and total weight zero, and with no vertices.
// Hence, each component of the weightedVertices and
// isVisited arrays is initialized as false.
public weightedDigraph()
{
 currSize = 0;
 currWeight = 0;
 for(int index = 0; index < ArraySize; ++index)
 {
  weightedVertices[index]= false;
  isVisited[index] = false;
  adjList[index] = new LinkedList();
 } // terminates text of for-loop
} // terminates text of constructor.
```

The instance methods `isEmpty()` and `size()` are defined and coded in exactly the same way as in `Digraph`. The method `totalWeight()` applies exclusively to weighted digraphs, and is implemented as

```
// Retrieves sum of the weights of all of the edges of
// the current weighted digraph.
public int totalWeight()
{
 return currWeight;
}  // terminates text of totalWeight().
```

The coding of the `insertVertex` method for weighted digraphs is the direct analog of its counterpart for (unweighted) digraphs. All that is done is the `boolean` value of the appropriate component of the `weightedVertices` array is changed from `false` to `true`. Otherwise, a `weightedDigraphException` exception is thrown.

```
// Inserts a new vertex if that vertex is not already present
// and raises an exception if no vertex is inserted, since it is
// already a vertex of the current weighted digraph.
public void insertVertex(weightedVertex p)
{
 // Vertex is not in present digraph.
```

```
// Perform legitimate insert operation.
if(!weightedVertices[p.getVertex()])
{
 ++currSize;
 weightedVertices[p.getVertex()] = true;
 vertexSet.add(new Integer(p.getVertex()));
} // terminates text of if-clause
else if((currSize < ArraySize) && weightedVertices[p.getVertex()])
// Vector is in current digraph.  Throw an exception.
 throw new weightedDigraphException("Vertex already in weighed
                                     digraph");
 else // overflow
  throw new weightedDigraphException("Overflow -- weighted digraph
                                     is full");
} // terminates text of insertVertex
```

The only difference between this version of `insertVertex` and its counterpart for `Digraph` is that the subscript of the vertex is computed in this new version by invoking the `getVertex()` method from the `weightedVertex` class.

The `insertEdge` method performs two operations. One is the actual insertion of the weighted edge in the current weighted digraph, assuming that the digraph does not already contain a weighted edge joining these vertices. The second accumulates the total weight of all of the edges by adding the new weight to the existing accumulated weight.

```
// Inserts a weighted edge from v to w.  Throws an exception
// if no edge is constructed.
// Precondition: v, w are vertices of the current weighted
// digraph, and w is not currently joined to v by a
// weighted edge.
public void insertEdge(weightedVertex v,weightedVertex w)
{
 // v, w are vertices of the current weighted digraph, and w is
 // not currently joined to v by a weighted edge.
 // Insert new weighted edge and accumulate total weight of
 // weighted digraph.
 if(weightedVertices[v.getVertex()] && weightedVertices[w.getVertex()]
      && !(adjList[v.getVertex()].contains(w)))
 {
  adjList[v.getVertex()].add(w);
  currWeight += w.getWeight();
 } // terminates text of if-clause
 else // if any other condition applies, throw new
      //  weightedDigraphException
  throw new weightedDigraphException("Illegal attempt to join weighted
                                      edge.");
} // terminates text of insertEdge().
```

The next method returns a `boolean` value, and tests whether the weighted vertex `v` is joined to the weighted vertex `w` by an edge. The method returns `true` if so, and `false` if not.

```
// Returns whether v is joined to w by a weighted edge.
// Returns true if so, false if not.
```

```
public boolean isAdjacent(weightedVertex v,weightedVertex w)
{
 return (adjList[v.getVertex()].contains(w)
                ||adjList[w.getVertex()].contains(v));
} // terminates text of isAdjacent().
```

The next method is the counterpart for weighted digraphs of `eraseEdge` of (unweighted) digraphs. The key difference now is that weighted edges are involved: the weight of the edge removed is deducted from the total weight of the digraph.

```
 // Removes weighted edge from v to w, if present in the current
 // weighted digraph.
 // Precondition: v, w are vertices in the current weighted digraph,
 // and there is a weighted edge from v to w.
 public void eraseEdge(weightedVertex v,weightedVertex w)
 {
  // There is a weighted edge from v to w, and each of v, w
  // is a weighted vertex in the current weighted digraph.
  if(isAdjacent(v,w) && weightedVertices[v.getVertex()]
      && weightedVertices[w.getVertex()])
  {
   adjList[v.getVertex()].remove(w);
   currWeight -= w.getWeight();
  } // terminates text of if-clause
  else // throw a weightedDigraphException exception
   throw new weightedDigraphException("Edge removal aborted");
 } // terminates text of eraseEdge().
```

The coding of the method `eraseVertex()` is left as an exercise. It is the analog for weighted digraphs of `eraseVertex()` for (unweighted) digraphs. The idea is to remove all edges that are incident to the weighted vertex to be removed. This implies that if `v` is the vertex to be removed, then each weighted edge that is adjacent to `v`, along with the weight attached to that edge, must be removed. In addition, `weightedVertices[v.getVertex()]` has to be set equal to `false`.

The final instance method to be considered is `output()`, used to output the specifications of the current weighted digraph object. This includes outputting the vertices, edges, and the total weight of the current object.

```
// Output specifications of current weighted digraph.
 public void output()
 {
  System.out.println("Vertices are:" + vertexSet);
  //for(int w = 0; w < ArraySize; ++w)
  // if(weightedVertices[w])
  //  System.out.print(w +"  ");
  // Terminates text of for-loop.
  System.out.println();
  System.out.println("Total weight is " + totalWeight());
  if(isEmpty())
   System.out.println("Current weighted digraph is empty.");
  else
   System.out.println("Current weighted digraph has "
```

```
                        + size() + " vertices");
} // Terminates text of output()
```

If we then execute the code sequence

```
weightedDigraph wDgraph1 = new weightedDigraph();
weightedVertex a0 = new weightedVertex(0,0);
wDgraph1.insertVertex(a0);
weightedVertex a1 = new weightedVertex(1,2);
wDgraph1.insertVertex(a1);
wDgraph1.insertEdge(a0,a1);
weightedVertex a2 = new weightedVertex(2,4);
wDgraph1.insertVertex(a2);
wDgraph1.insertEdge(a1,a2);
wDgraph1.insertEdge(a2,a0);
weightedVertex a3 = new weightedVertex(0,2);
weightedVertex a4 = new weightedVertex(1,4);
weightedVertex a5 = new weightedVertex(4,2);
wDgraph1.insertVertex(a5);
wDgraph1.insertEdge(a5,a0);
wDgraph1.output();
```

the output is

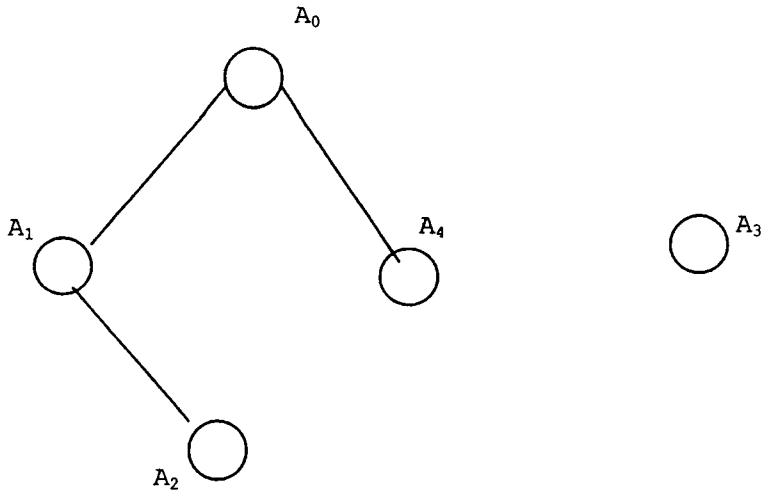
```
Vertices are [0, 1, 2, 4]
Total weight is 8
Current weighted digraph has 4 vertices
```

The start vertex is `0`. Since it is the first vertex inserted, there is no second vertex available to form an edge. We must nevertheless assign a weight component for this vertex which is consistent with maintaining the total weight. Therefore, we assign a weight of zero; consequently, the initial construction of the start vertex is

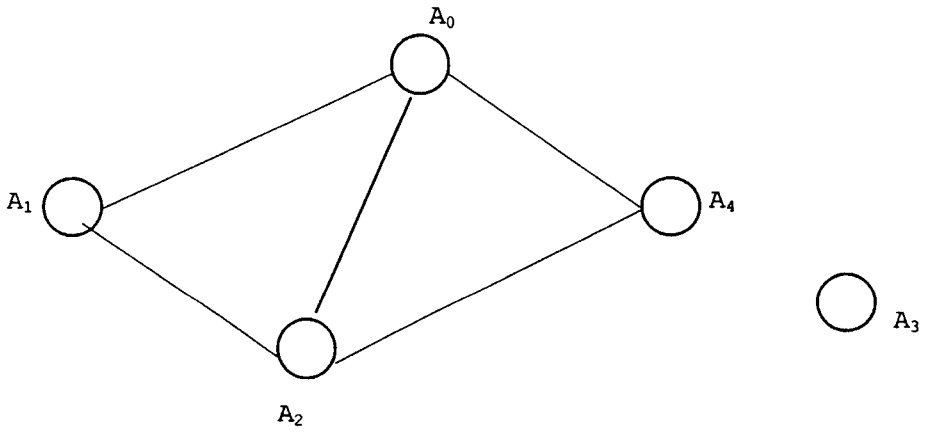
```
weightedVertex a0 = new weightedVertex(0,0);
```

3. Spanning Trees.

Let `G = (V,E)` be an undirected graph. Then the graph `H = (V',E')` is called a subgraph of `G` if $V' \subseteq V$ and $E' \subseteq E$, that is, if every vertex of `H` is also a vertex of `G` and every edge of `H` is also an edge of `G`. For example, the graph described in (Figure 4) is a subgraph of the graph of (Figure 5):

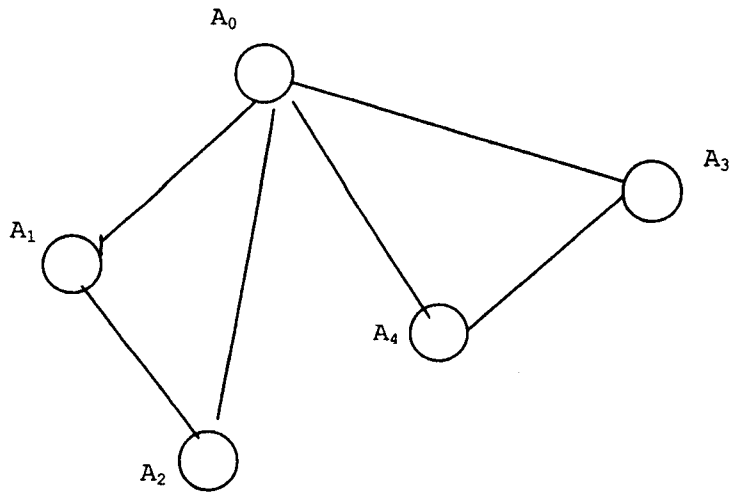


(Figure 4)



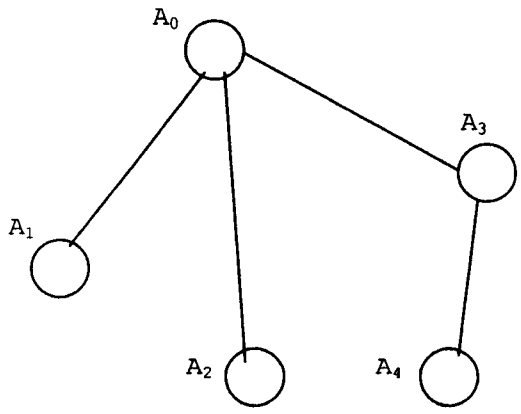
(Figure 5)

A subgraph of G that contains all of the vertices of G and is a tree is called a spanning tree of G. For example, suppose G is the connected graph described in (Figure 6(a)):

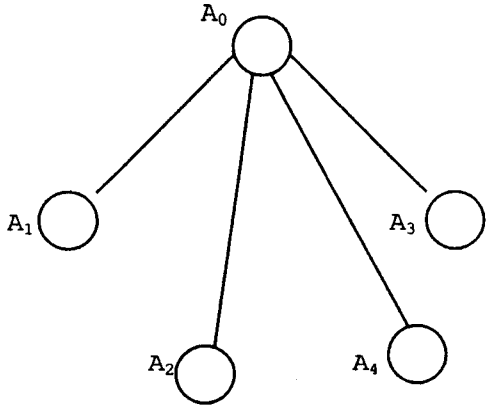


(Figure 6(a))

Then each of the following is a spanning tree for G:



(Figure 6(b))



(Figure 6(c))

Finding a spanning tree for a connected graph has important applications, since such trees contain the smallest number of possible edges, and still retain the connection links between all of the vertices of the underlying graph. Specifically, if the graph is connected and has `n` vertices, then we can show that the spanning tree contains `n - 1` edges. You may verify this for the graph of (Figure 6(a)) and the spanning trees of (Figure 6(b)) and (Figure 6(c)).

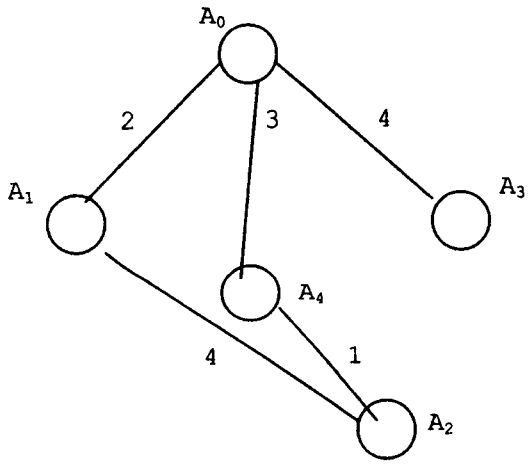
Suppose we turn to weighted connected graphs. These have very important applications, since, for example, such graphs may model maintenance costs between nodes in a communications network, or the travel costs between cities. In the implementation of weighted digraphs discussed in Section 2, we showed how we may accumulate the weight (that is, the cost) for every edge in a weighted digraph. It is not difficult to describe a similar operation for accumulating the weight in an undirected graph. We can then determine a spanning tree that yields a minimum accumulated weight. The spanning tree with the least weight is called a minimal spanning tree for that graph.

Suppose `G = (V,E)` is a connected weighted graph. How do we find a minimal spanning tree for `G`? There are two algorithms that produce a minimal spanning tree: Prim's algorithm and Kruskal's algorithm. We begin by looking at Prim's algorithm.[3] We begin by thinking of `V` as subdivided into two disjoint subsets: `In` (for "Inside") = those elements of `V` already included in the minimal spanning tree constructed so far, and `Out` (for "Outside") = those elements in `V` not appearing in the minimal spanning tree constructed so far. Initially, `In` is the set consisting of only the start vertex, and `Out` is the set consisting of the remaining elements of `V`. The objective is to add new elements to `In` as they are removed from `Out`, in succession, by including a new edge joining any element of `In` to an element of `Out` containing the least possible weight. The process terminates when `In = V` and `Out = ∅`, the empty set. The set `V` then contains the vertices of the minimal spanning tree, and the edges that are adjoined at each stage represent the edges of the minimal spanning tree.

Prim's algorithm is an example of a greedy algorithm, namely, an algorithm that is based on the selection of a finite collection of objects to join to an initial collection in increments by iteration. Each cycle of the iteration always chooses some object that minimizes some cost function. In the case of Prim's algorithm, the cost function is defined as a function that joins an element of `In` to an element of `Out` by an edge of minimal weight.

Example 2. Suppose we consider the weighted graph of (Figure 7):

[3] Prim, R. C., "Shortest Connection Networks and Some Generalizations," Bell System Technical Journal, Vol. 36 (1957), pp. 1389-1401.

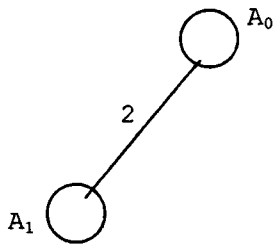


(Figure 7)

Begin with $In = \{A_0\}$, $Out = \{A_1, A_2, A_3, A_4\}$, and E' = current set of edges of the minimal spanning tree = $\varnothing$.

(1) The edges connected to A_0: (A_0, A_1) has weight 2,
(A_0, A_3) has weight 4,
(A_0, A_4) has weight 3.

Thus, add A_1 to In, and add (A_0, A_1) to E'. The minimal spanning tree so far consists of

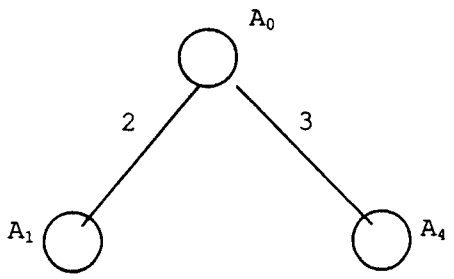


(Figure 8(a))

(2) Add edges connecting A_1 to previous edges not selected:

(A_1, A_2) has weight 4,
(A_0, A_3) has weight 4,
(A_0, A_4) has weight 3.

Thus, add A_4 to In and (A_0, A_4) to E'. The minimal spanning tree so far consists of

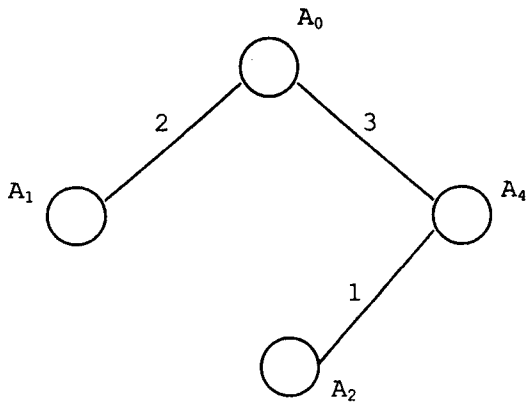


(Figure 8(b))

(3) Add edges containing A_4 to previous edges not selected:

(A_4, A_2) has weight 1,
(A_1, A_2) has weight 4,
(A_0, A_3) has weight 4.

Thus, add A_2 to In and (A_4, A_2) to E'. The minimal spanning tree so far consists of

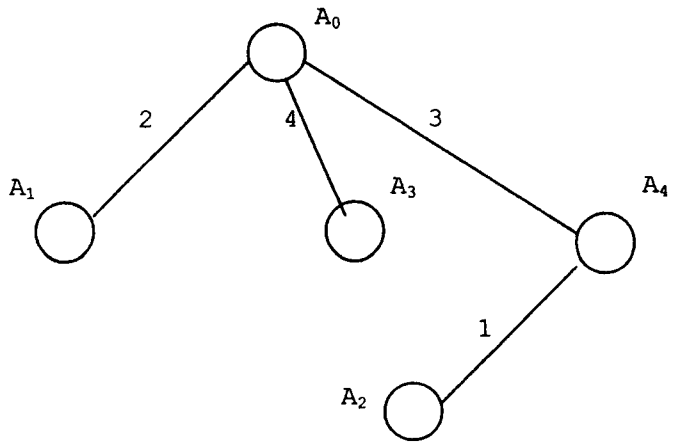


(Figure 8(c))

(4) Add edges containing A_2 to previous edges not selected:

(A_1, A_2) has weight 4,
(A_0, A_3) has weight 4.

Thus, add A_3 to In and (A_0, A_3) to E'. Since now In = V, the desired minimal spanning tree is



(Figure 8(d))

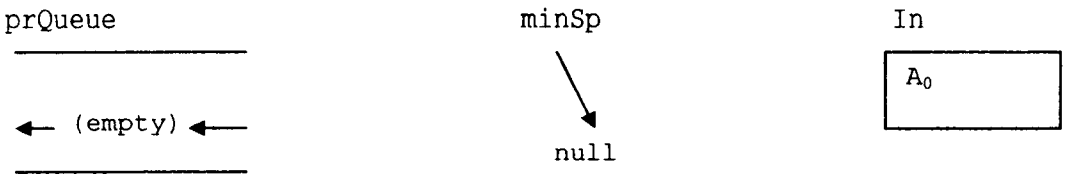
Prim's algorithm may be designed with the connected weighted graph `G` as the only parameter. The algorithm begins by constructing an initially empty tree `minSp`[4] and an initially empty priority queue `prQueue`. The role of `minSp` will be to keep track of that portion of the minimal spanning tree constructed so far, and `prQueue` will store the edges of the graph in order of increasing weight. In addition, a set of vertices `In` will store the number of vertices of `G` that are currently in `minSp`. Initially, `In` will consist only of A_0, the designated start vertex of `G`. The algorithm may then be written as

Initialize `minSp` *as empty;*
Initialize `prQueue` *as empty;*
Initialize `In = [A0];`
Set `currVertex = A0;`
`while(V != In)` `// V` *is the set of all vertices of* `G`
{
Insert all edges containing `currVertex` *into* `prQueue;`
// This insertion may be done using the `for`*-loop*
`//` *for (all edges* `e` *containing* `currVertex`)
`//  prQueue.add(e);`
// Then choose edge at front of current `prQueue`
`e = prQueue.front();`
// Remove edge from `prQueue`
`prQueue.remove();`
// Insert `e` *as new edge of* `minSp`
`minSp.insert(e);`
// Add `currVertex` *to* `In`
`In.add(currVertex);`
} *// terminates text of* `while`-loop.
// At this point `V == In` *and* `minSp` *is the desired*

[4] Note that `minSp` is not necessarily a binary tree.

```
// minimal spanning tree. Return minSp.
return minSp;
```

We illustrate the algorithm on the graph of Example 2. Initially, `minSp` and `prQueue` are empty and `In` contains only A_0, the start vertex of `G`. See (Figure 9(a)):

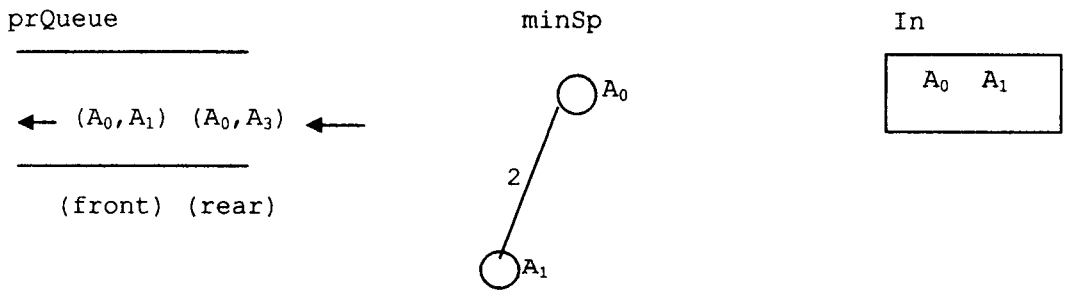


(Figure 9(a))

Processing enters the `while`-loop for the first time, since `V != In`. Thus, each of (A_0,A_1), (A_0,A_3), (A_0,A_4) is added to `prQueue` in the order from front to rear as

(A_0,A_1) (A_0,A_4) (A_0,A_3)

Then `e` = (A_0,A_1) is removed from `prQueue`, `currVertex` is set at A_1, `e` is inserted into `minSp`, and the value of `currVertex` is added to `In`, yielding the results indicated in (Figure 9(b)):

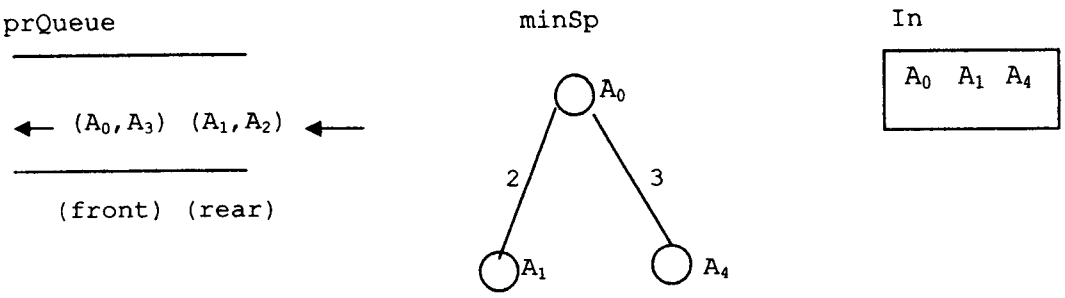


(Figure 9(b))

Now `V != In` is still true, so processing re-enters the `while`-loop. Here we add any new edges containing A_1 into `prQueue`. This involves a single edge (A_1,A_2). When this edge is inserted into its proper position, `prQueue` is read from front to rear as

(A_0,A_4) (A_0,A_3) (A_1,A_2)

Set e = (A_0, A_4), remove e from prQueue, set currVertex = A_4, insert e into minSp, and the value of currVertex is added to In, producing the result seen in (Figure 9(c)):

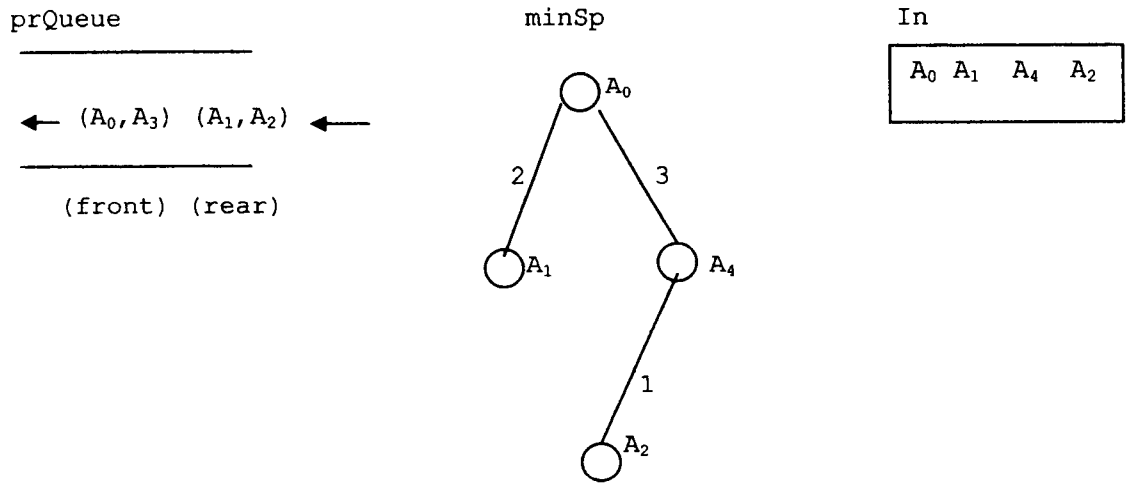


(Figure 9(c))

This completes another cycle of the while-loop. Since V != In is still true, processing re-enters the while-loop. Since the value of currVertex is A_4, we add all new edges containing A_4 to prQueue. This involves the single edge (A_4, A_2), and thus prQueue reads from front to rear as

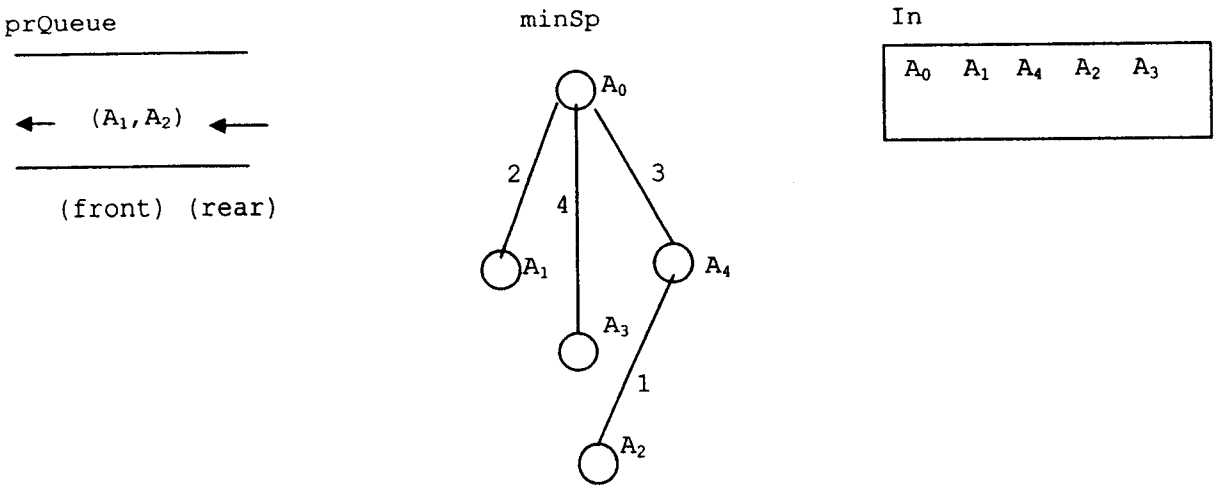
(A_4, A_2) (A_0, A_3) (A_1, A_2)

Set e = (A_4, A_2), remove e from prQueue, set currValue = A_2, insert e into minSp, and add currValue to In, producing the results seen in (Figure 9(d)):



(Figure 9(d))

This completes the execution of another cycle of the `while`-loop. Since `V != In` still holds, processing re-enters the `while`-loop. But now since there is no edge in `G` joining A_2 to A_3, `prQueue` does not change. Instead, `e` becomes (A_0, A_3) and is removed from `prQueue`, (A_0, A_3) is added to `minSp`, and `currVertex` = A_3 is added to `In`, producing the results of (Figure 9(e)):



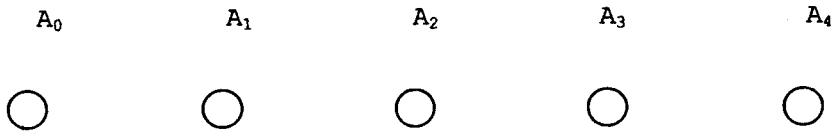
(Figure 9(e))

This completes another cycle of the `while`-loop. At this point, `V != In` is false; therefore, control passes to the return of the current version of `minSp`. This is the desired minimal spanning tree of `G`.

To analyze Prim's algorithm, we note that its implementation involves an outer `while`-loop that executes `n` times, where `n` is `V.size()`, the number of vertices of `G`. But each such cycle involves an inner `for`-loop, which executes once for each edge containing the current value of `currVertex`. If we assume that there are `m` edges, where $m \leq n - 1$ since `G` is connected, we conclude that Prim's algorithm is or order `O(m*n)`.

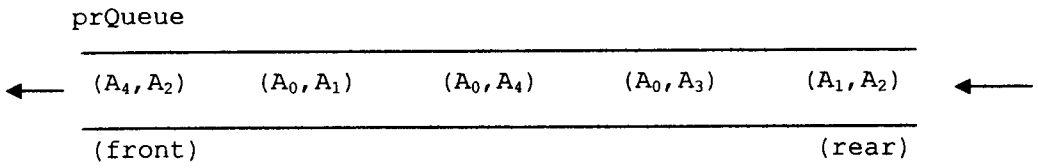
Kruskal's algorithm[5] is another example of a greedy algorithm. It takes a completely different approach in constructing the minimal spanning tree of `G` by joining subtrees and edges of `G` until two subtrees remain. These are then joined by an edge to form the minimal spanning tree. To illustrate Kruskal's algorithm, we again consider the graph of Example 2, and begin by observing that each vertex of `G` may be viewed as a subtree with no edges, as in (Figure 10(a)):

[5] Kruskal, Joseph B., "On the Shortest Spanning Tree of a Graph and the Traveling Salesman Problem," Proceedings of the American Mathematical Society, Vol. 7 (1956), pp. 48-50.



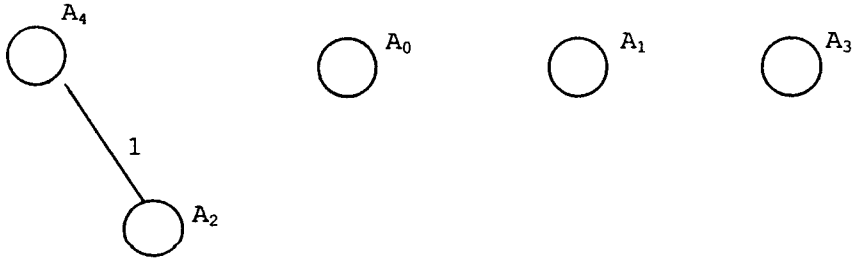
(Figure 10(a))

Configurations of weighted subtrees are commonly called (weighted) forests. Order the edges of `G` by inserting all of the edges in a priority queue `prQueue` in order of increasing weight (see (Figure 10(b)):



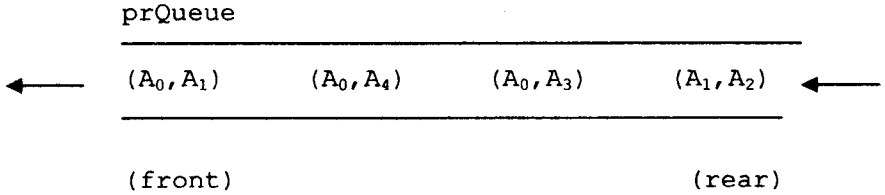
(Figure 10(b))

Remove edge (A_4, A_2) from `prQueue`, creating a new forest as described in (Figure 10(c)):



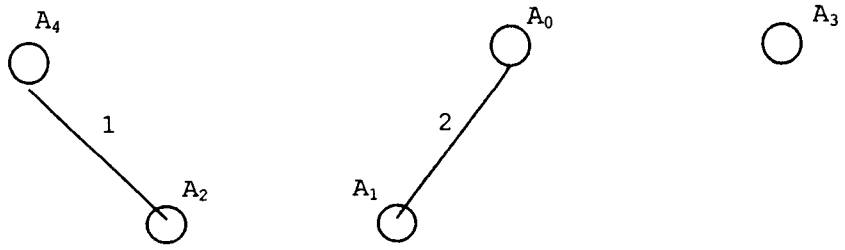
(Figure 10(c))

with `prQueue` now given by



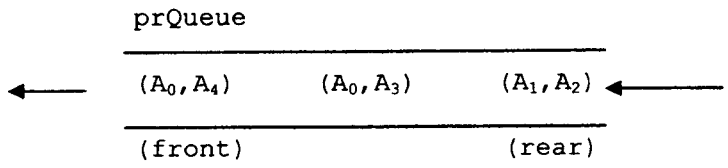
(Figure 10(d))

Remove (A_0, A_1) from prQueue, creating a new forest



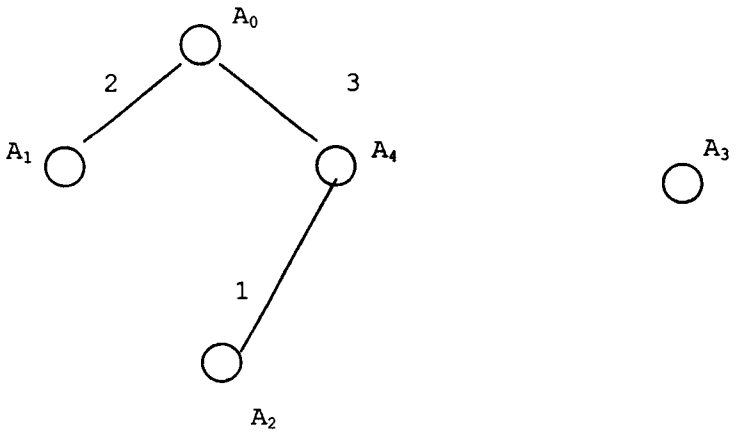
(Figure 10(e))

with prQueue given by



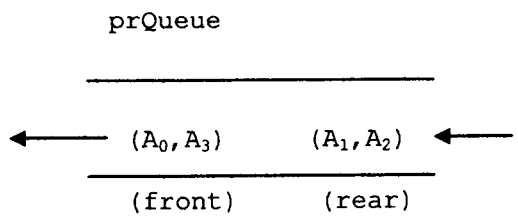
(Figure 10(f))

Remove edge (A_0, A_4) from prQueue, creating a new forest



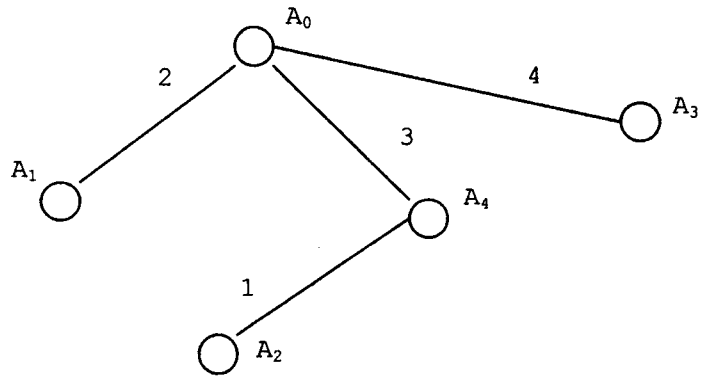
(Figure 10(g))

with accompanying prQueue



(Figure 10(h))

Now remove edge (A_0, A_3) from prQueue, creating the forest consisting of the single weighted tree



(Figure 10(I))

This is the desired minimal spanning tree for G.

How do we express Kruskal's algorithm in pseudocode? As in Prim's algorithm, G is passed as the only parameter. However, unlike Prim's algorithm, minSp is initialized as the forest whose members are just the elements of V, and prQueue is initialized as the members of E, ordered according to increasing weight. The algorithm continues execution by successively removing weighted edges from prQueue, joining these among the members of minSp until minSp contains exactly one tree, the desired minimal spanning tree of G. Thus the pseudocode may be expressed as

```
Initialize minSp = V;
Initialize prQueue = E, ordered in order of increasing weight;
// Add new edges from prQueue to members of minSp
// to create a new forest.
for(int index = 1; index <= E.size() && minSp.size() <= V.size() - 1;
            ++index)
  if(prQueue.contains(edge[index])
    minSp.add(edge[index]);
```

```
// terminates text of for-loop
// At this point, all of the remaining edges have been joined
// and minSp consists of a single element. This is the
// minimal spanning tree for G. Return this tree.
return minSp;
```

The `add()` method inside the loop is invoked as many times as is necessary to join all of the existing vertices of `G` without producing a cycle. That is, as many times as is necessary so as to maintain members of `minSp` as trees. This operation is of order `O(n)`, where `n = V.size()`, the number of vertices of `G`. The construction of the initial version of `prQueue` requires a sort operation of order `O(m log m)`, where `m = E.size()`, the number of edges of `G`. Consequently, using the fact that `m ≤ n - 1` (see the Exercises at the end of this chapter), this yields the following result, estimating the complexity of Kruskal's algorithm.

If `G` is a connected weighted graph with `n` vertices and `m` edges, Kruskal's algorithm constructs a minimal spanning tree for `G`, with a complexity of `O(m log n)`.

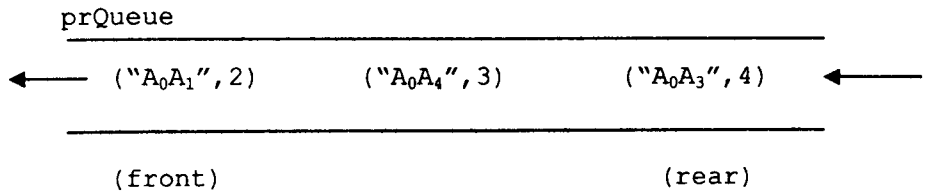
4. Shortest Paths. Dijkstra's Algorithm.

A very important and practical application of the theory of weighted graphs involves finding the shortest path between two vertices. We define the shortest path between two vertices A_i and A_j of a weighted graph `G` to be the path with the least weight between A_i and A_j. This may not necessarily be the path with the least number of vertices joining A_i and A_j.

There is a greedy algorithm, due to Dijkstra, for finding the shortest path. This algorithm was referred to earlier in this paper. We describe the algorithm here, and refer to it as the Shortest Path Algorithm. The solution proposed by Dijkstra involves a design using a pair of nested loops, and is thus of order $O(n^2)$.

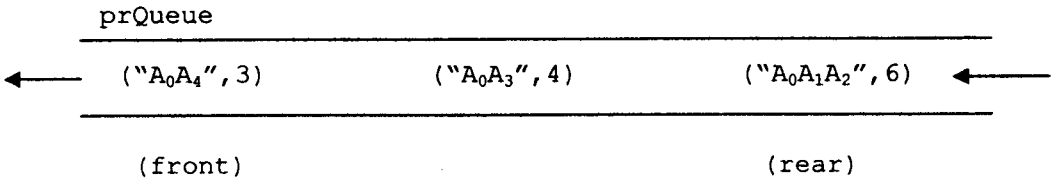
The algorithm involves a priority queue `prQueue` of pairs: the first component is a character string describing a path beginning with the initial vertex and ending with the vertex described so far toward the destination vertex. The second component of the pair is the accumulated weight of the path described by the first component. In addition, the pairs appearing in the priority queue will be arranged in order of increasing weight.

We illustrate this algorithm using the weighted graph `G` of (Figure 1). Suppose we wish to determine the shortest path from A_0 to A_2. We begin by initializing `prQueue` by inserting all edges beginning at A_0 as in (Figure 11(a)):



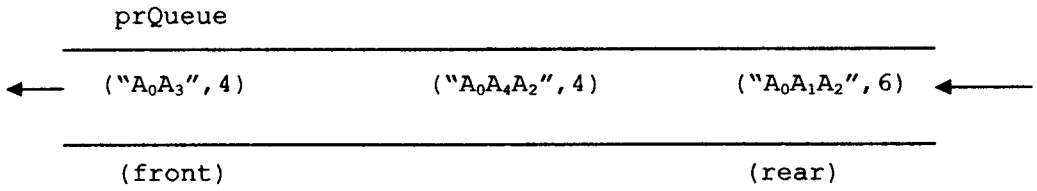
(Figure 11(a))

After this initialization, the pair ("A_0A_1", 2) is removed from prQueue, and we iterate over the neighbors of A_1. The only other vertex of an edge from A_1 is A_2, and the weight of that edge is 4. Accumulate the weight of the path $A_0A_1A_2$ to 6, and insert the pair ("$A_0A_1A_2$", 6) into prQueue as shown in (Figure 11(b))



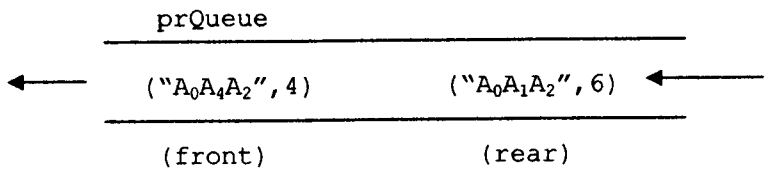
(Figure 11(b))

Again remove the pair ("A_0A_4", 3) from the front of prQueue, and iterate over the neighbors of A_4. The vertex A_2 is a neighbor of A_4, and the edge joining A_4 to A_2 has a weight of 1. Accumulate the weight of the path $A_0A_4A_2$ to 4, and insert ("$A_0A_4A_2$", 4) into prQueue. See (Figure 11(c)).



(Figure 11(c))

Remove ("A_0A_3", 4) from prQueue. Since there is no edge from A_3 to A_2, we do not insert any pair back into prQueue. Since pathCount is zero, the front of the current prQueue, as depicted in (Figure 11(d)) contains the shortest path.



(Figure 11(d))

We describe a pseudocode version of Dijkstra's algorithm. To do so, we must make a number of preliminary observations. First, for ease of computation, we will represent a vertex of the graph by its integer-valued subscript. For example, the vertex A_0 will be represented by `0`, the vertex A_1 by `1`, and so on. Secondly, we introduce the `int`-valued variable `numPaths`, whose current value is the number of distinct paths from the designated source vertex to the destination vertex. Indeed, if we consider the weighted graph of (Figure 7), and if the source vertex is `0` (for A_0) and if the destination vertex is `2` (for A_2), then `numPaths` is `2`.

We also define a `Pair` class whose objects represent pairs of the form `(path,weight)`, where `path` is a `String` variable whose characters are integer-valued digits describing a path from the source vertex to some vertex of the graph, and where `weight` is the integer-valued weight of that path. As an example, the `Path` object `("012",6)` defines the path $A_0A_1A_2$ with total weight `6` from the weighted graph described in (Figure 7). The formal class definition of `Pair` is

```
public class Pair
{
 // Constructor
 public Pair(String pth,int wt)
 {
  path = pth;
  weight = wt;
 } // terminates text of constructor
 // Instance methods
 // Retrieve path
 public String getPath()
 {
  return path;
 } // terminates text of getPath
 // Returns weight of current Pair object
 public int getWeight()
 {
  return weight;
 } // terminates text of getWeight
 // Instance variables
 private String path;
 private int weight;
}// terminates text of Pair class
```

In addition, we introduce a `boolean`-valued method `edgeExists` defined for any two vertices `u,v` by

$$\texttt{edgeExists(u,v)} = \begin{cases} \texttt{true, if there is an edge of the graph joining u,v} \\ \texttt{false, if not.} \end{cases}$$

Thus, if we refer to the weighted graph of (Figure 7), `edgeExists(4,2) = true`, and `edgeExists(3,2) = false`. We may implement `edgeExists` using either the adjacency matrix representation of the weighted graph (see (Figure 2)) or the adjacency list representation of the weighted graph (see (Figure 3)).

Finally, we define the integer-valued variable `numVertices` as the variable storing the number of vertices of the weighted graph.

The pseudocode for Dijkstra's algorithm may then be expressed as

```
public Pair getShortestPath(int source,int destination,int numPaths)
{
 // pathCount stores the number of paths from source to destination
 int pathCount = numPaths;
 // Construct priority queue of Pair objects
 ArrPriorityQueue<Integer> prQueue = new ArrPriorityQueue<Integer>();[6]
 // Initialize prQueue with all Pair objects representing edges
 // joining the designated source vertex to any vertex of the graph
 // connected by an edge.
 for(int index = 0; index < numVertices;++index)
  if(edgeExists(source,index))
   prQueue.add(Pair object with first component consisting of edge
               represented by a string from source to index, and
               second component equal to the weight of that edge);
 // Continue path to destination vertex
 while(pathCount> 0)
 {
  // Retrieve front pair stored in prQueue
  Pair frontPair = (Pair)prQueue.front();
  // Remove that front Pair object
  prQueue.remove();
  for(int index = 0; index < numVertices; ++index)
   if(edgeExists(last vertex of frontPair,index))
   {
    frontPair.path += index converted to String;
    frontPair.weight += getWeight(Pair object with first component last
                                  vertex, index, weight);
    // Insert new version of frontPair into prQueue
    prQueue.add(new Pair(frontPair.path,frontPair.weight));
   } // terminates text of if-clause
   // also terminates text of for-loop
  // Decrement value of pathCount
  if(newVertex == destination)
```

[6] This can be replaced by the predefined `PriorityQueue` implementation of Java 5.0.

```
  --pathCount;
 } // terminates text of while-loop
 // Retrieve Pair object at front of current prQueue
 Pair frontPair = (Pair)prQueue.front();
 // Remove any unwanted Pair object occupying position at front
 // of current prQueue.
 if((pathCount == 0) && (last vertex of frontPair != dstination))
  prQueue.remove();
 else  // front of prQueue contains desired shortest path
  return (Pair)prQueue.front();
 // If there is no Pair from source to destination, return null
 return null;
} // terminates text of getShortestPath
```

There is a variant of Dijkstra's algorithm due to R. W. Floyd,[7] that solves the problem of finding the shortest path between any two vertices in any graph `G` whose edges have positive weights. The idea behind the algorithm is very simple, but its complexity is $O(n^3)$ for a graph of `n` vertices, since three nested loops are needed for its implementation. The algorithm begins by setting up a two-dimensional array whose components represent the shortest distance of paths joining the vertices given by the row and column subscripts. Initially, the components of this array has values given by the components of the corresponding adjacency matrix representation of the weighted graph. In the case of a component with an initial value of ∞, we instead use a large positive integer value supported by Java's primitive `int` type. The algorithm then proceeds by searching for shorter distances between each pair of vertices. The resulting array then contains components representing the shortest distance between any vertices of the weighted graph.

To give a formal description of the algorithm, we define `weight[][]` as the array whose components are the respective weights of the edges of the underlying weighted graph. We also define `shortest[][]` as the array whose respective components are the values of the shortest paths between the associated vertices. Thus, the algorithm may be expressed as

```
public void allPairsShortestPaths(int[][] weight,int[][] shortest)
{
 // Initialize shortest array
 for(int index1 = 0; index1 < n; ++index1)
  for(int index2 = 0; index2 < n; ++index2)
   shortest[index1][index2] = weight[index1][index2];
 // Find shortest path between any two vertices
 for(int index1 = 0; index1 < n; ++index1)
  for(int index2 = 0; index2 < n; ++index2)
   for(int index3 = 0; index3 < n; ++index3)
    if(shortest[index2][index1] + shortest[index1][index3]
         < shortest[index2][index3])
     shortest[index2][index3] = shortest[index2][index1]
                              + shortest[index1][index3];
} // terminates text of allPairsShortestPaths
```

[7] Floyd, R. W., "Algorithm 97: Shortest path," Communications of the Association for Computing Machinery, vol. 5, no. 6, 1962, p. 345.

The complexity of $O(n^3)$ is a consequence of the three nested `for`-loops used to compute the shortest path between any two vertices. As an example, if we consider the graph of (Figure 7), `weight[][]` is represented as[8]

	0	1	2	3	4
0	10000	2	10000	4	3
1	2	10000	4	10000	10000
2	10000	4	10000	10000	1
3	4	10000	10000	10000	10000
4	3	10000	1	10000	10000

(Figure 12)

The final form of `shortest[][]` is

	0	1	2	3	4
0	4	2	4	4	3
1	2	4	4	6	5
2	4	4	2	8	1
3	4	6	8	8	7
4	3	5	1	7	2

(Figure 13)

[8] Here "∞" is represented by 10000.

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[3] Floyd, R. W., "Algorithm 97: Shortest Path," *Communications of the Association for Computing Machinery*, Vol. 5, No. 6, 345.

[4] Kruskal, Joseph B., "On the Shortest Spanning Tree of a Graph and the Traveling Salesman Problem," *Proceedings of the American Mathematical Society*, Vol. 7 (1956), 48-50.

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